String effective actions, dualities, and generating solutions

Chemissany, Wissam Ali

Appendix A

Elementary Differential Geometry

In this appendix we will bring together the ideas of differential geometry, which are required throughout the thesis. There are several books, e.g, [171, 172], written for physicists, which explore the subject at greater length and greater depth.

A.1 Convention

Apart from chapter 3 which has its own convention, we take the following metric signature in the main text to be

$$g = \mathrm{diag}(- \cdots -, + \cdots +), \tag{A.1.1}$$

with (-) occurring t times and (+1) occurring s times. The pair (s,t) is called the signature of the metric g.

A.2 Introductory Concepts

A.2.1 Manifolds

A D-dimensional manifold is a topological space together with a family of open sets M_i that cover it, i.e, $M = \bigcup_i M_i$. M_i's are called coordinate patches. Within one patch one may defines a 1:1 map ϕ_i, called the chart, from $M_i \to \mathbb{R}$. Concretely speaking, a point $p \in M_i \subset M$ is mapped to $\phi_i(p) = (x^1, x^2, \cdots, x^D)$. We say that the set $(x^1, x^2, \cdots, x^D)$ are the local coordinates of the point p in the patch M_i.
If $p \in M_i \cap M_j$, then the map $\phi_j(x'^1, x'^2, \cdots, x'^D)$ provides a second set of coordinates

for the point p. The composite map

$$\phi_i \circ \phi_j : \qquad \mathbb{R}^D \to \mathbb{R}^D \tag{A.2.1}$$

is then specified by the set of functions $x'^\mu(x^\nu)$. These functions, and there inverses $x^\nu(x'^\mu)$ are required to be smooth, usually C^∞.

A.2.2 Tensor Fields

Scalar, Vector Fields and 1-Forms

The simplest object to define on a manifold M are scalar functions f that map $M \to \mathbb{R}$. We say that the point p maps to $f(p) = z$. On each coordinate patch M_i we can define the compound map $f \circ \phi_i^{-1}$ from $\mathbb{R}^D \to \mathbb{R}$ as $f_i(x^\mu) \equiv f \circ \phi_i^{-1}(x^\mu) = z$. On the overlap $M_i \cap M_j$ of two patches with local coordinate x^μ and x'^ν of the point p, the two descriptions of f must agree. Thus $f_i(x^\mu) = f_j(x'^\nu)$.

Vectors on a manifold M always describe tangents vectors to a curve in M. Let $p(t)$ be some curve. The coordinates of this curve are $x^i(p(t))$, $i = 1 \cdots D$ and the tangent vector to the curve is given by $\frac{d}{dt}x^i(p(t))$. Defining the differential operator

$$X = X^i \frac{\partial}{\partial x^i}, \qquad \text{with } X^i = \frac{dx^i(p(t))}{dt} \tag{A.2.2}$$

we obtain

$$\frac{d}{dt} f(p(t)) = Xf, \tag{A.2.3}$$

where f is a function on M. The tangent space to the manifold M at p, the space of all possible tangents at p, is denoted by $T_p(M)$.

To the contravariant vectors, which we have considered up to now, there also exist their duals - the covariant vectors. The dual space to $T_p(M)$ is the cotangent space $T_p^*(M)$ where duality is defined via the inner product $(dx^i, \frac{\partial}{\partial x^j}) = \delta^i{}_j$.
An element of $T_p^*(M)$ is given by the so-called 1-form

$$w = w_i dx^i \in T_p^*(M), \tag{A.2.4}$$

where $\{dx^i\}$ represents the dual basis in $T_p^*(M)$.

Tensor Algebra

We can now construct tensors of type (a, b) by mapping a elements of $T_p^*(M)$ and b elements of $T_p(M)$ into $\mathbb{R}$. So the space of these tensors is defined by

$$T^a{}_b = \underbrace{T_p(M) \otimes \cdots \otimes T_p(M)}_{a \text{ factors}} \otimes \underbrace{T_p^*(M) \otimes \cdots \otimes T_p^*(M)}_{b \text{ factors}}. \tag{A.2.5}$$

In terms of local coordinates, it reads[1]

$$T(x) = T^{i_1 \cdots i_a}_{j_1 \cdots j_b} \partial_{i_1} \cdots \partial_{i_a} dx^{j_1} \cdots dx^{j^b} \in T^a{}_b. \tag{A.2.6}$$

The action of T on 1-forms $w_1, \cdots, w_a$ and vectors $X_1, \cdots, X_b$ gives the number

$$T(w_1, \cdots, w_a, X_1, \cdots, X_b) = T^{i_1 \cdots i_a}_{j_1 \cdots j_b} w_{1i_1} \cdots w_{ai_a} X_1^{j_1} \cdots X_b^{j_b}. \tag{A.2.7}$$

Allowing the point p to vary smoothly over the whole manifold, the vectors and tensors also vary smoothly over M, and one achieves so-called **vector fields** and **tensor fields** on M.

We now introduce the additional structure of a **metric** on a manifold. A metric or an inner product on a real vector space V is a non-degenerate bilinear map on each $V \otimes V \to \mathbb{R}$. The inner product of two vectors $u, v \in V$ is a real number denoted by (u, v). The inner product must satisfy the following properties:

(i) bilinearity -$(u, c_1 v_1 + c_2 v_2) = c_1(u, v_1) + c_2(u, v_2)$.

(ii) non-degeneracy- if $(u, v) = 0$ for all $v \in V$, then $u = 0$.

(iii) symmetry-$(u, v) = (v, u)$.

A metric on a manifold is a smooth assignment of a inner product on each $T_p(M) \otimes T_p(M) \to \mathbb{R}$. In a local coordinates the metric is specified by a covariant second rank tensor field $g_{\mu\nu}$ whose determinant denoted by g, and the inner product of two vectors fields U^μ and V^μ is $g_{\mu\nu} U^\mu V^\nu$, which is a scalar field. In particular the metric gives the length τ of a curve $x^\mu(t)$ with tangent vector dx^μ/dt.

Specifying a metric on a manifold, it will help with the classification of manifolds. In other words, the manifold is said to be **Riemannian** if its metric satisfies the following axioms at each point $p \in M$;

(i) $g(U, V) = g(V, U)$,

(ii) $g(U, U) \geq 0$, equality only for $U = 0$.

This means the metric evaluated at point p is a symmetric positive definite bilinear form. A **pseudo-Riemannian** manifold is a manifold endowed with a metric which obeys, beside axiom (i), the axiom (ii') states that if $g(U, V) = 0$ for all $U \in T_p M$, then $V = 0$, i.e., the manifold has an indefinite signature.

Differential forms: With the help of the wedge product

$$dx^\mu \wedge dx^\nu := dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu, \tag{A.2.8}$$

[1]The Einstein convention is used throughout the text; any index that appears twice in an expression is summed over if it appears once as upper index and once as a lower index.

one can define now several differential forms

$$0-\text{form} \qquad w = w(x) \tag{A.2.9}$$

$$1-\text{form} \qquad w = w_\mu dx^\mu \tag{A.2.10}$$

$$p-\text{form} \qquad w = \frac{1}{p!} w_{\mu_1 \cdots \mu_p}(x) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu p}. \tag{A.2.11}$$

We denote the set of all p-forms by Λ^p. This is a vector space of dimension[2]

$$\dim \Lambda^p = \binom{D}{p} = \frac{D!}{p!(D-p)!}. \tag{A.2.12}$$

One can therefore construct $(p+q)$-forms out of p-forms and q-forms in a straightforward manner by means of the wedge product $\alpha_p \wedge \beta_q \in \Lambda^{p+q}$, in such a way that

$$\alpha_p \wedge \beta_q = \chi_{p+q} \Rightarrow \chi_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!q!} \alpha_{[\mu_1 \cdots \mu_p} \beta_{\mu_{p+1} \cdots \mu_{p+q}]}. \tag{A.2.13}$$

Commuting the forms α_p and β_q, one also obtains

$$\alpha_p \wedge \beta_q = (-)^{pq} \beta_q \wedge \alpha_p. \tag{A.2.14}$$

All forms belong to the space

$$\Lambda^* = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \cdots \Lambda^D, \tag{A.2.15}$$

which is closed under the wedge product operation (or exterior product). Λ^* is a graded algebra, also named Cartan's exterior algebra (Grassmann algebra).

One differentiates the forms by introducing the **exterior derivative**, namely $d = \partial_\mu dx^\mu$, acting on a p-form in the following way

$$dw = \frac{1}{p!} \partial_\nu w_{\mu_1 \cdots \mu_p}(x) dx^\nu \wedge \cdots \wedge dx^{\mu_p}. \tag{A.2.16}$$

In fact, the exterior product is a map d: $\Lambda^p \to \Lambda^{p+1}$ which transforms p-forms into (p+1)-forms, satisfying nilpotency condition $d^2 = 0$ as well as obeying the antiderivation rule

$$d(\alpha_p \wedge \beta_q) = (d\alpha_p \wedge \beta_q) + (-)^p \alpha_p \wedge d\beta_q. \tag{A.2.17}$$

A p-form that satisfies $d\alpha_p$=0 is called **closed**. A p form α_p that can be expressed as $\alpha_p = d\alpha_{p-1}$ is called **exact**. Poincaré's lemma implies that locally any closed p-form can be expressed as $d\alpha_{p-1}$, but α_{p-1} may not be well defined globally on M.

[2] Note that Λ^p and Λ^{D-p} have the same dimensions.

Trace, (anti)-commutator: Next we define the trace, the commutator and the anti-commutator of differential forms.

Let $\alpha_p \in V \otimes \Lambda^p$, $\beta_q \in V \otimes \Lambda^q$ be forms which are V-valued, where V is actually a linear vector space consisting of vectors, e.g., Lie algebra or matrices. One says V-valued $\alpha_{\mu_1 \cdots \mu_p}, \beta_{\nu_1 \cdots \nu_q} \in V$

$$\alpha_{\mu_1 \cdots \mu_p} = \alpha^i_{\mu_1 \cdots \mu_p} T_i \tag{A.2.18}$$

$$\beta_{\mu_1 \cdots \mu_q} = \beta^i_{\mu_1 \cdots \mu_q} T_i, \tag{A.2.19}$$

with T_i the vectors (generators, matrices) of a vector space V. This definition actually means the direct product, like $\alpha = T_i \otimes \alpha^i \in V \otimes \Lambda^p$, between the basis $\{T_i\}$ of V and the wedge of differential forms.

Since, for instance, T^i matrices satisfy $[T_i, T_j] = f_{ijk} T^k$, where f^{ijk} are the anti-symmetric structure constants, one can derive the following rules;

$$[\alpha_p, \beta_q] = \alpha_p \wedge \beta_q - (-)^{pq} \beta_q \wedge \alpha_p = -(-)^{pq} [\beta_q, \alpha_p] \tag{A.2.20}$$

$$\{\alpha_p, \beta_q\} = \alpha_p \wedge \beta_q + (-)^{pq} \beta_q \wedge \alpha_p = (-)^{pq} \{\beta_q, \alpha_p\} \tag{A.2.21}$$

$$[\alpha_p \wedge \beta_q, \gamma_r] = \alpha_p \wedge [\beta_q, w_r] + (-)^{pr} [\alpha_p, w_r] \wedge \beta_q \tag{A.2.22}$$

$$\mathrm{tr}(\alpha_p \wedge \beta_q) = (-)^{pq} \mathrm{tr}(\beta_q \wedge \alpha_p), \qquad \text{also } \mathrm{tr}[\alpha_p, \beta_q] = 0. \tag{A.2.23}$$

Hodge $\star$ operation: The fact that the space of all p-forms Λ^p and the space Λ^{D-p} have the same dimensions, implies a duality between 2 spaces, an isomorphism given by the Hodge $\star$ operation; $\Lambda^p \xrightarrow{\star} \Lambda^{D-p}$. In other words, the star $\star$ transforms p-forms into $(D-p)$-forms and its action is defined by

$$\alpha_{\mu_1 \cdots \mu_{D-p}} = \frac{1}{p!} \epsilon_{\mu_1 \cdots \mu_{D-p}}{}^{\nu_1 \cdots \nu_p} \beta_{\nu_1 \cdots \nu_p}, \tag{A.2.24}$$

and denoted by $\star\beta_p$. The natural choice of ϵ is specified up to sign, i.e. up to a choice of the orientation, by the condition

$$\epsilon^{\mu_1 \cdots \mu_D} \epsilon_{\mu_1 \cdots \mu_D} = (-)^s D!, \tag{A.2.25}$$

with s the number of minuses appearing in the signature of the metric $g_{\mu\nu}$. It is also worth noting that

$$\epsilon^{\mu_1 \cdots \mu_D} \epsilon_{\nu_1 \cdots \nu_D} = (-)^s D! \delta^{[\mu_1}{}_{\nu_1} \delta^{\mu_2}{}_{\nu_2} \cdots \delta^{\mu_D]}{}_{\nu_D}. \tag{A.2.26}$$

Contraction of equation A.2.25 over j indices yields

$$\epsilon^{\mu_1 \cdots \mu_j \mu_{j+1} \cdots \mu_D} \epsilon_{\mu_1 \cdot \mu_j \nu_{j+1} \cdots \nu_D} = (-)^s j! (D-j)! \delta^{[\mu_{j+1}}{}_{\nu_{j+1}} \delta^{\mu_2}{}_{\nu_2} \cdots \delta^{\mu_D]}{}_{\nu_D}. \tag{A.2.27}$$

The totally antisymmetric ϵ-tensor or **Levi-Cevita tensor** is precisely defined by

$$\epsilon_{\mu_1 \cdots \mu_D} = \begin{cases} (-)^\sigma & \text{if all } \mu_i \text{ are distinct} \\ 0 & \text{otherwise,} \end{cases} \tag{A.2.28}$$

where σ is the signature of the permutation $(1, \cdots, D) \to (\mu_1, \cdots, \mu_D)$.
Note that

$$\epsilon^{\mu_1 \cdots \mu_D} = g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \cdots g^{\mu_D \nu_D} \epsilon_{\nu_1 \nu_2 \cdots \nu_D} = g^{-1} \epsilon_{\mu_1 \mu_2 \cdots \mu_D}, \tag{A.2.29}$$

and

$$\epsilon_{12 \cdots D} = [(-1)^s \det(g_{\mu\nu})]^{1/2} = \sqrt{|g|}. \tag{A.2.30}$$

The inner product associated with the star $\star$ operation can, up to some integral over M, be written as follows

$$\alpha_p \wedge \star \beta_p = \frac{1}{p!} \alpha_{\mu_1 \cdots \mu_p} \beta^{\mu_1 \cdots \mu_p} \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^D, \tag{A.2.31}$$

with $\epsilon = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^D$ is the natural volume element of M. The action of star on $\star \beta_p$ yields

$$\star \star \beta_p = (-)^{p(D-p)+s} \beta_p. \tag{A.2.32}$$

A.3 Homogeneous Spaces, Isometries and Geodesic Flow

This section is based on a section in the book by Nakahara [171]. We will assume that the reader is familiar with Lie groups.

A.3.1 Homogeneous Spaces

Let us start by defining the action of a group on a manifold.
Definition: Given a Lie Group G and differentiable manifold M, we define an **action** of G on M to be a differentiable map σ: $G \times M \to M$, which satisfies the following conditions:

(i) $\sigma(e, p) = p$ for any $p \in M$,

(ii) $\sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p)$ for any $g_1 g_2 \in G$ and any $p \in M$,

where e is the identity element of the group.

We also need to define the following properties of the group actions:
Definition: Let G be a Lie group that acts on a manifold M by $\sigma : G \times M \to M$. The action σ is said to be

(a) **transitive** if, for any p_1, $p_2 \in M$, there exists an element $g \in G$ such that $\sigma(g, p_1) = p_2$;

(b) **free** if every non-trivial element $g \neq e$ of G has no fixed points in M. In other words, given an element $g \in G$, if there exists an element $p \in M$ such that $\sigma(g,p) = p$, then g must be the identity element e.

Now we are ready to define a homogeneous space. A manifold M is said to be **homogeneous**, if there exists a Lie group G that acts *transitively* on M. The n-sphere is homogeneous because its group $\mathrm{SO}(n+1)$ acts transitively on it.

Definition: Let G be a Lie group that acts on a manifold M. The **isotropy** group of $p \in M$ is a subgroup of G defined by

$$H(p) = [g \in G | \sigma(g,p) = p]. \tag{A.3.1}$$

This means that $H(p) \subset G$ is the group of elements that leave p fixed. This is called the *little group or stabilizer*. If G acts transitively on M, one can show that isotropy groups of all points in M are isomorphic to each other.

Theorem: Under certain conditions, if one has a homogeneous manifold M with the group acting on it with isotropy group H, then the coset space G/H is a manifold (i.e. it has a differentiable structure), and it is diffeomorphic to M, i.e. $G/H \cong M$. As an example we have $\mathrm{SO}(n+1)/\mathrm{SO}(n) \cong S^n$.

M is said to be isotropic at p if all tangent vectors at p can be rotated into each other by elements of the isotropy group of p. This matches our intuition that isotropy means that space 'looks' the same in every direction. Spaces that are homogeneous and isotropic are said to be *maximally symmetric*.

A.3.2 Isometries, Geodesic Flow

Isometry

An isometry of a manifold (M, g) is a diffeomorphism[3] $f : M \to M$ which preserves the metric

$$f^* g_{f(p)} = g_p \qquad \text{or} \quad g_{f(p)}(f_* U, f_* V) = g_p(U, V), \tag{A.3.2}$$

where f^* and f_* are respectively the pullbacks and the push-forwards of f. In components one can write A.3.2

$$g_{\mu\nu}(p) = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)), \tag{A.3.3}$$

where x^μ and y^α are respectively the coordinates of p and $f(p)$. If we take the infinitesimal isometry to be generated by ϵX, the vector field X is called the *Killing vector field*. This leads to the following *Killing equation*

$$X^\rho \partial_\rho g_{\mu\nu} + \partial_\mu X^\sigma g_{\sigma\nu} + \partial_\nu X^\lambda g_{\mu\lambda} = 0. \tag{A.3.4}$$

[3]Diffeomorphism is an invertible function that maps one manifold to another, such that both the function and its inverse are smooth.

As an example we consider D-dimensional Minkowski spacetime ($D \geq 2$) there exist $D(D+1)/2$ Killing vector fields, D of which generate the translations, $(D-1)$ boosts and $(D-1)(D-2)/2$ space rotations. Such spaces which admit $D(D+1)/2$ Killing vector fields are example of the maximally symmetric spaces defined above.

Geodesic Flow

A vector field on a manifold M describes, quite naturally, a **flow** in M. We consider the integral curve $\sigma(t,x)$ of a vector field $U \in T_x(M)$ passing through x at a time $t = 0$. In a given patch one has

$$\frac{d\sigma^\mu(t,x)}{dt} = U^\mu(\sigma(t,x)) \qquad \text{with } \sigma(0,x^\mu) = x^\mu. \tag{A.3.5}$$

Such an integral curve representing a map $\sigma : \mathbb{R} \times M \to M$, is termed a flow generated by the vector U.

A geodesic defined with respect to a connection on a manifold M gives the local extremum of the length of an integral curve connecting two points. Let $c : (a,b) \to M$ be a curve in M. If the tangent vector $U(t)$ on $c(t)$ is parallel transported along $c(t)$, namely if

$$\nabla_U U = 0 \tag{A.3.6}$$

the integral curve $c(t)$ is called **geodesic**, i.e. the straightest as well as the shortest possible curve, where ∇ is the covariant derivative defined below. In components, the geodesic equation A.3.6 becomes

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0, \tag{A.3.7}$$

where $\{x^\mu\}$ are the local coordinate of $c(t)$ and $\Gamma^\mu{}_{\nu\rho}$ is the connection coefficients.

The parameter t typically represents time for a timelike curve, or distance for a spacelike curve. This parameter cannot be chosen arbitrarily. Rather, it must be chosen so that the tangent vector dx^μ/dt has a constant magnitude. This is referred to as an **affine parametrization**. Any two affine parameters are linearly related. That is, if r and t are affine parameters, then there exist constants a and b such that $r = at + b$.

A.4 Connections, Curvatures and Covariant Derivatives

The (pseudo)-Riemannian manifold (M, g) that physicists use in General Relativity is a D-dimensional spacetime endowed with a bilinear form, $(2,0)$ tensor with signature

$(-\cdots-,+\cdots+)$, taking on the form

$$ds^2 = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu, \qquad \mu,\nu = 1,\cdots D. \tag{A.4.1}$$

The Levi-Cevita connection following from this metric is

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} - \partial_\nu g_{\mu\sigma} + \partial_\sigma g_{\mu\nu}\right), \tag{A.4.2}$$

from which one obtain the Riemann tensor

$$\mathcal{R}^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\gamma_{\nu\sigma}\Gamma^\mu_{\gamma\rho} - \Gamma^\gamma_{\nu\rho}\Gamma^\mu_{\gamma\sigma}. \tag{A.4.3}$$

The Ricci tensors $\mathcal{R}_{\nu\rho}$ and the Ricci scalar $\mathcal{R}$ are defined via the contractions as follows

$$\mathcal{R}_{\nu\sigma} \equiv \mathcal{R}^\mu{}_{\nu\mu\sigma}, \qquad \mathcal{R} \equiv \mathcal{R}^\mu_\mu. \tag{A.4.4}$$

In addition, the Einstein tensor $G_{\mu\nu}$ takes on the form

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}. \tag{A.4.5}$$

The action of the covariant derivative ∇, associated to the general coordinate transformation, on a (p,q) tensor is defined by

$$\begin{aligned}\nabla_\alpha T^{\nu_1\cdots\nu_q}_{\mu_1\cdots\mu_p} &= \partial_\alpha T^{\nu_1\cdots\nu_q}_{\mu_1\cdots\mu_p} - \Gamma^\rho_{\alpha\mu_1}T^{\nu_1\cdots\nu_q}_{\rho\mu_2\cdots\mu_p}\cdots - \Gamma^\rho_{\alpha\mu_p}T^{\nu_1\cdots\nu_q}_{\mu_1\cdots\cdots\mu_{p-1}\rho}\\ &+ \Gamma^{\nu_1}_{\rho\alpha}T^{\rho\nu_2\cdots\nu_q}_{\mu_1\cdots\mu_p} + \cdots + \Gamma^{\nu_q}_{\rho\alpha}T^{\nu_1\cdots\nu_{q-1}\rho}_{\mu_1\cdots\mu_p}.\end{aligned} \tag{A.4.6}$$

The action of the box operator $\Box$ on a scalar field Φ is given by

$$\Box\Phi = \nabla_\mu\partial^\mu\Phi = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\Phi\right), \tag{A.4.7}$$

where g is the determinant of $g_{\mu\nu}$.

One can prove that for maximally symmetric space the Riemann tensor is expressed as

$$\mathcal{R}_{\rho\sigma\mu\nu} = C(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}), \tag{A.4.8}$$

where C is a constant. In the metric Ansatz 6.2.2, g_{ab} often describes an Euclidean maximally symmetric space. This means we have the sphere S^n for $k=1$, the hyperboloid $\mathbb{H}^n$ for $k=-1$ or flat space $\mathbb{E}^n$ for $k=0$. Then we have

$$ds^2 = \frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2_{n-1}, \tag{A.4.9}$$

where $d\Omega^2_m$ is the metric on the S^m sphere defined by

$$d\Omega^2_m = d\theta_1^2 + \sin^2(\theta_1)d\theta_2^2 + \cdots + \sin^2(\theta_1)\cdots\sin^2(\theta_{m-1})d\theta_m^2. \tag{A.4.10}$$

Performing the following redefinition

$$\frac{1}{1-kr^2}dr^2 = d\eta^2, \tag{A.4.11}$$

one obtain the metrics

$$\begin{aligned} k=-1: &\qquad ds^2 = d\eta^2 + \sinh^2\eta d\Omega^2_{n-1}, \\ k=0\ \ : &\qquad ds^2 = d\eta^2 + \eta^2\, d\Omega^2_{n-1}, \\ k=+1: &\qquad ds^2 = d\eta^2 + \sin^2\eta\, d\Omega^2_{n-1}. \end{aligned} \tag{A.4.12}$$

The Ricci tensor corresponding to these metrics can be obtained by having $C = k$, namely $\mathcal{R}_n = kn(n-1)$.

Appendix B

Some Calculational Details for Chapter 3

In this appendix we will give some calculational details related to section 3.4 of chapter 3 . Most of the conventions and notations will be used in this appendix are the same of [100]. The parameter α which will appear throughout this appendix is a free parameter proportional to α', the inverse of string tension.

B.1 Lagrangian Density and Redefinitions

In [100] the Lagrangian density behaves

$$\mathcal{L}_R = \frac{1}{2}e\phi^{-3}\left(-R(\omega) - \frac{3}{2}\widetilde{H}_{\mu\nu\rho}\widetilde{H}^{\mu\nu\rho} + 9(\phi^{-1}\partial_\mu\phi)^2\right), \tag{B.1.1}$$

with the following definitions:

$$\begin{aligned}
\widetilde{H}_{\mu\nu\rho} &= \partial_{[\mu}B_{\nu\rho]} - \alpha\sqrt{2}\mathcal{O}_{3\,\mu\nu\rho}(\Omega_-),\\
\mathcal{O}_{3\,\mu\nu\rho} &= \Omega_{-[\mu}{}^{ab}\partial_\nu\Omega_{-\rho]}{}^{ab} - \frac{2}{3}\Omega_{-[\mu}{}^{ab}\Omega_{-\nu}{}^{ac}\Omega_{-\rho]}{}^{cb},\\
\Omega_{-\mu}{}^{ab} &= \omega_\mu^{ab} - \frac{3}{2}\sqrt{2}\widetilde{H}_\mu{}^{ab}.
\end{aligned} \tag{B.1.2}$$

Antisymmetrasation brackets are with weight 1.

First we redefine the field in order to make the comparison of the actions tractable. The redefinitions are:

1. The dilaton changes as

$$\phi^{-3} \to e^{-2\Phi}, \qquad (\phi^{-}\partial\phi) \to \frac{2}{3}\partial\Phi. \tag{B.1.3}$$

2. For the two and three-form, one can set

$$\widetilde{H} \to \frac{1}{3\sqrt{2}}\widetilde{H}, \qquad B \to \frac{1}{\sqrt{2}}B. \tag{B.1.4}$$

The Lagrangian $\mathcal{L}_R$ then becomes

$$\mathcal{L}_R = \frac{1}{2}ee^{-2\Phi}\left(-R(\omega) - \frac{1}{12}\widetilde{H}_{\mu\nu\rho}\widetilde{H}^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi\right) \tag{B.1.5}$$

as in 3.4.4.

The spin connections $\omega(e)$ solve the equations

$$\mathcal{D}_\mu e_\nu{}^a - \mathcal{D}_\nu e_\mu^a = 0, \qquad \text{with } \mathcal{D}_\mu e_\nu{}^a \equiv \partial_\mu e_{\nu\,a} - \omega_\mu{}^{ac} e_{\nu\,c}. \tag{B.1.6}$$

The Riemann tensor and related quantities are defined as

$$R_{\mu\nu}{}^{ab}(\omega) = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu\,c}{}^b + \omega_\nu{}^{ac}\omega_{\mu\,c}{}^b\,, \tag{B.1.7}$$

$$R_\mu{}^a(\omega) = e^\nu{}_b R_{\mu\nu}{}^{ab}(\omega)\,, \tag{B.1.8}$$

$$R(\omega) = e^\mu{}_a R_\mu{}^a(\omega)\,. \tag{B.1.9}$$

B.2 Equations of Motion

The lowest order equations of motion, i.e., at order α'^0 are:

$$\mathcal{S} = ee^{-2\Phi}[R(\omega) - 4\mathcal{D}_a\partial^a\Phi + 4(\partial_a\Phi)^2 + \frac{1}{2}H^{abc}H_{abc}] = 0, \tag{B.2.1}$$

$$\mathcal{B}^{\nu\rho} = \frac{1}{4}\partial_\mu(ee^{-2\Phi}H^{\mu\nu\rho}) = 0, \tag{B.2.2}$$

$$\mathcal{E}^\lambda{}_c = -\frac{1}{2}e^\lambda{}_c\mathcal{S} + ee^{-2\Phi}(R^\lambda{}_c(\omega) + \frac{1}{4}(H^2)^\lambda{}_c - 2e^\lambda{}_d\mathcal{D}_c\Phi\partial^d\Phi) = 0. \tag{B.2.3}$$

In the main text of section 3.4 we use a field redefinition to eliminate any contribution proportional to the Ricci tensor. The required equation is, modulo $\mathcal{E}$ and $\mathcal{S}$:

$$R_\mu{}^a(\omega) = 2\mathcal{D}_\mu\partial^a\Phi - \frac{1}{4}(H^2)_\mu{}^a. \tag{B.2.4}$$

B.3 Expanding $\mathcal{L}_R$ in Powers of α

The 3-form field $\widetilde{H}$ is defined recursively by (3.4.6, 3.4.7, 3.4.8). We find

$$\widetilde{H}_{\mu\nu\rho} = H_{\mu\nu\rho} - 6\alpha(\mathcal{O}_{3\,\mu\nu\rho}(\omega) + \mathcal{A}_{\mu\nu\rho}) = \bar{H}_{\mu\nu\rho} - 6\alpha\mathcal{A}_{\mu\nu\rho}, \tag{B.3.1}$$

where $\mathcal{O}_{3\,\mu\nu\rho}$ is the gravitational contribution (order α^0) of the Lorentz Chern-Simons term, and

$$\begin{aligned}\mathcal{A}_{\mu\nu\rho} = &\frac{1}{2}\partial_{[\mu}(\omega_\nu{}^{ab}\widetilde{H}_{\rho]}{}^{ab}) - \frac{1}{2}R^{ab}_{[\mu\nu}(\omega)\widetilde{H}_{\rho]}{}^{ab} + \frac{1}{4}\widetilde{H}_{[\mu}{}^{ab}\mathcal{D}_\nu\widetilde{H}_{\rho]}{}^{ab}\\ &+ \frac{1}{12}\widetilde{H}_{[\mu}{}^{ab}\widetilde{H}_\nu{}^{ac}\widetilde{H}_{\rho]}{}^{cb}.\end{aligned} \tag{B.3.2}$$

To order α $\mathcal{L}_R$ B.1.5 can be expressed as

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}ee^{-2\Phi}[-R(\omega) - \frac{1}{12}\bar{H}_{\mu\nu\rho}\bar{H}^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi\\ &+ \alpha\{\frac{1}{2}H^{\mu\nu\rho}\partial_\mu(\omega_\nu{}^{ab}H_\rho{}^{ab}) - \frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)H_\rho{}^{ab}H^{\mu\nu\rho} + \frac{1}{4}H^{\mu\nu\rho}H_\mu{}^{ab}\mathcal{D}_\nu H_\rho{}^{ab} +\\ &+ \frac{1}{12}H^{\mu\nu\rho}H_\mu{}^{ab}H_\nu{}^{ac}H_\rho{}^{cb}\}].\end{aligned} \tag{B.3.3}$$

The term with $H\partial(\omega H)$ is after partial integration, proportional to B.2.2 and can be eliminated by a field redefinition.

B.4 Simplification of $\mathcal{L}_{R^2}$ Terms

We often use the identity

$$\mathcal{D}_{[a}(\Omega_-)\widetilde{H}_{bcd]} = -\frac{3}{2}\alpha R_{[ab}{}^{ef}(\Omega_-)R_{cd]}{}^{ef}(\Omega_-), \tag{B.4.1}$$

to isolate terms that are of higher order terms in α. The term 3.4.14 can be simplified by using the cyclic identity for the Riemann tensor:

$$R_{\mu\nu}{}^{ab}(\omega)H^{\mu\,ac}H^{\nu\,cb} = -\frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)H^{\mu\nu\,c}H^{abc}. \tag{B.4.2}$$

Now we consider 3.4.15. Note that the two terms written in 3.4.15 are actually the same. Then we have

$$\begin{aligned}\frac{1}{2}(\mathcal{D}_\mu H_\nu{}^{ab} - \mathcal{D}_\nu H_\mu{}^{ab})H^{\mu ac}H^{\nu cb} &= -\mathcal{D}_\mu H_\nu{}^{ab}H^{\mu\,ac}H^{\nu bc}\\ &= -\mathcal{D}_{[e}H_{fab]}H^{eac}H^{fbc}.\end{aligned} \tag{B.4.3}$$

The term is completely of order α'^2. Finally we consider 3.4.13. This can be rewritten as

$$\frac{1}{2}ee^{-2\Phi}(\mathcal{D}_\mu H_\nu{}^{ab} - \mathcal{D}_\nu H_\mu{}^{ab})\mathcal{D}^\mu H^{\nu ab} = ee^{-2\Phi}(2R_{\mu\nu}{}^{ab}H^{\mu ac}H^{\nu cb} + R_\mu{}^c H^{\mu ab}H_{abc}$$
$$+ e^\mu{}_c e^\nu{}_d \mathcal{D}_\nu H_{abd}\mathcal{D}_\mu H_{abc} + 2\partial_c\Phi H_{abd}\mathcal{D}_d H_{abc} - 2\partial_d\Phi H_{abd}\mathcal{D}_c H_{abc}$$
$$+ 2\mathcal{D}_c H_{abd}\mathcal{D}_{[c}H_{abd]}). \tag{B.4.4}$$

The last term is of order α'^2.

Appendix C

Some Lie Algebra Theory

In this introductory appendix we define the basic concepts relating to Lie group and Lie algebra [131, 173, 174].

C.1 Classical Lie Groups

Consider a group G acting on a space V over a field F, e.g. ($\mathbb{R}$ or $\mathbb{C}$). We can think of G as being matrices, and of V as a vector space on which these matrices act. A group element $g \in G$ transforms the vector $v \in V$ into $g \cdot v = v'$.

Once an additional structure, in the form of a metric, has been imposed on an N-dimensional vector space over a field F, one would be able to classify the classical (matrix) groups acting on V. Recall the definition of the metric

$$(v_1, v_2) = f \qquad v_1,\, v_2 \in V,\, f \in F \tag{C.1.1}$$

obeying the following conditions:

$$(v_1, av_2 + bv_3) = a(v_1, v_2) + b(v_1, v_3) \tag{C.1.2a}$$

and

$$(av_1 + bv_2, v_3) = (v_1, v_3)a + (v_2, v_3)b \tag{C.1.2b}$$

or

$$(av_1 + bv_2, v_3) = (v_1, v_3)a^* + (v_2, v_3)b^* \tag{C.1.2c}$$

Metrics obeying conditions C.1.2a and C.1.2b are called *bilinear* metrics; those obeying C.1.2a and C.1.2c are called sesquilinear. The groups metric-preserving are then classified as follows[1]

[1] Orthogonal groups preserving metric (p, q) in ($\mathbb{R}$ or $\mathbb{C}$) are denoted by $\mathrm{O}(p, q, \mathbb{R})$, $\mathrm{O}(p, q, \mathbb{C})$.

(a) Groups preserving *bilinear* symmetric metrics are called **orthogonal**.

(b) Groups preserving *bilinear* antisymmetric metrics are called **symplectic.**

(c) Groups preserving *sesquilinear* symmetric metrics are called **unitary**.

The metric preserving group which are in addition volume-preserving are called the special metric-preserving groups and are denoted by an additional S., e.g., SO(n), Sp(n), and SU(n).

In addition, we have five isolated groups, which are called

$$E_6, \qquad E_7, \qquad E_8, \qquad G_2, \qquad F_4. \tag{C.1.3}$$

In all groups the subscript denotes the rank of the group. Those five isolated groups are referred to as the **exceptional** Lie groups.

C.2 Structure of Simple Lie algebra

C.2.1 The basics

A complex Lie algebra $\mathfrak{G}$ is a vector space over F endowed with a binary operation which is called a Lie bracket commutator

$$[\,,] : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}. \tag{C.2.1}$$

The two defining properties of $[\,,]$ read

$$[X, X] = 0 \qquad \forall\, X \in \mathfrak{G} \tag{C.2.2}$$

and

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \qquad \forall\, X,\, Y,\, Z \in \mathfrak{G}. \tag{C.2.3}$$

The identity C.2.3 is the so-called **Jacobi identity.**
A Lie algebra is specified by a set of generators $\{T^a\}$ and their commutator relations

$$[T^a, T^b] = f^{bc}{}_d T^d, \tag{C.2.4}$$

where $f^{bc}{}_d$ are the structure constants. The *dimension* d of the lie algebra $\mathfrak{G}$ is thus the dimension of the underlying vector space spanned by the basis

$$\mathcal{B} = \{T^a | a = 1, \cdots, d\}. \tag{C.2.5}$$

Simple Lie algebras are Lie algebras which contain no proper[2] ideal and which is not abelian. An ideal or invariant subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ is a subspace satisfying simultaneously $[\mathfrak{H},\mathfrak{H}] \subseteq \mathfrak{H}$ and $[\mathfrak{H},\mathfrak{G}] \subseteq \mathfrak{H}$. An **abelian** Lie algebra is a Lie algebra which satisfies $[\mathfrak{G},\mathfrak{G}] = 0$. A direct sum of simple Lie algebras forms the so-called **semi-simple** Lie algebra.

Levi Theorem: Every Lie algebra can be decomposed into the direct sum of simple Lie algebra and solvable algebras; solvable Lie algebra can be defined iteratively by the series $\mathfrak{s}^0 = \mathfrak{s}$, $\mathfrak{s}^1 = [\mathfrak{s}^0, \mathfrak{s}^0]$, $\mathfrak{s}^i = [\mathfrak{s}^{i-1}, \mathfrak{s}^{i-1}]$, for a finite number of steps, it ends up with zero.

In general the action of a Lie algebra $\mathfrak{G}$ on a vector space V is carried out via a linear **representation** of $\mathfrak{G}$

$$R: \ \mathfrak{G} \to \mathfrak{gl}(\mathfrak{G}): \ X \to R(X), \qquad R(X): \ V \to V : \ v \to R(X) \cdot v. \tag{C.2.6}$$

It is possible to represent $\mathfrak{G}$ on itself; thereby one obtains the **adjoint** representation: for any $T \in \mathfrak{G}$

$$\mathrm{ad}T(T_a) \equiv [T, T_a], \qquad \Rightarrow [\mathrm{ad}T_a]^c{}_b = -f_{ab}{}^c. \tag{C.2.7}$$

Exponentiating the generators of the Lie algebra $\mathfrak{G}$ in the adjoint representation, we get the adjoint representation of the corresponding group G

$$\mathrm{Ad}(g) = \exp[\tau^a \mathrm{ad}(T_a)], \qquad \text{with } T'_b = T_a[\mathrm{Ad}(g)]^a{}_b, \tag{C.2.8}$$

where $g \in G$ and τ is the group parameter. Actually, in any representation R, the adjoint action of G on $\mathfrak{G}$ is given by

$$R(g)R(T_a)R(g^{-1}) = R(T_b)[\mathrm{Ad}(g)]^b{}_a. \tag{C.2.9}$$

The Killing metric $B(.\,,.)$ is a symmetric bilinear form defined by

$$B(T_a, T_b) \equiv \mathrm{tr}(\mathrm{ad}T_a \mathrm{ad}T_b) = f_{ac}{}^d f_{bd}{}^c. \tag{C.2.10}$$

Suppose the Lie algebra is semisimple[3]. According to Cartan's criterion, *the Killing metric is non-degenerate for a semisimple algebra*. This means $\det B_{ab} \neq 0$, so that the inverse of B_{ab}, denoted by B^{ab}, exists. Since the Killing metric is also real and symmetric, it can be reduced, choosing an orthonormal basis, to canonical form $B_{ab} = \mathrm{diag}(-1, \cdots, -1, 1, \cdots, 1)$ with p (-1's) and $(d-p)$ (+1's) are respectively the number of compact and non-compact generators (see next section), where d is the dimension of $\mathfrak{G}$. When thinking about a real form, that will be discussed in the next section, its convenient to visualize it in terms of the signature of its metric.

In any (semi)-simple Lie algebra $\mathfrak{G}$ there are two kinds of generators: there is a

[2] Any Lie algebra has two subalgebras, namely $\mathfrak{G}$ itself and zero. These subalgebras are called trivial subalgebras; any other subalgebra of $\mathfrak{G}$ is called proper subalgebra of $\mathfrak{G}$.

[3] This is true for all classical Lie algebras except for the Lie algebras $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{u}(n, \mathbb{C})$.

maximal abelian subalgebra, called the **Cartan subalgebra** CSA $\mathfrak{h} = H_1, \cdots, H_r$, $[H_I, H_J] = 0$ for two elements of CSA. There are shift operators denoted by E_α. α is an r-dimensional vector $\alpha = (\alpha_1, \cdots, \alpha_r)$ and r is the **rank** of $\mathfrak{G}$. The latter are eigenoperators of the H_I in the adjoint represenation belonging to α_I: $[H_I, E_\alpha] = \alpha_I E^\alpha$. For each eigenvalue, or **roots** α^I, there is another eigenvalue $-\alpha_I$ and a corresponding eigenoperator $E_{-\alpha}$ under the action of H_I.

Suppose we represent each element of the Lie algebra by an $n \times n$ matrix. Then $[H_I, H_J] = 0$ means that the matrices H_I can all be diagonalized simultaneously. The eigenvalues β_I are given by $H_I|\beta\rangle = \beta_I|\beta\rangle$, where the eigenvectors are labelled by the **weight vector** $\beta = (\beta_1, \cdots, \beta_r)$. The canonical commutation relations are summarized by :

$$[H_I, H_J] = 0, \qquad [H_I, E_\alpha] = \alpha_I E_\alpha, \qquad [E_\alpha, E_{-\alpha}] = \alpha_I H_I. \tag{C.2.11}$$

C.2.2 Real Forms

Let us recall some definitions,
$V^{\mathbb{C}}$: Let V be a vector space over $\mathbb{R}$. $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ is called the *complexification* of V. One has $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{\mathbb{C}}$.
$W^{\mathbb{R}}$: Let W be a vector space over $\mathbb{C}$. Restricting the definition of scalars to $\mathbb{R}$ then leads to a vector space $W^{\mathbb{R}}$ over $\mathbb{R}$ and $\dim_{\mathbb{C}} W = 1/2 \dim_{\mathbb{R}} W^{\mathbb{R}}$.

Real form of $\mathfrak{G}^{\mathbb{C}}$: Let $\mathfrak{G}^{\mathbb{C}}$ be a Lie algebra over $\mathbb{C}$. A **real form** of $\mathfrak{G}^{\mathbb{C}}$ is a subalgebra $\mathfrak{G}$ of the real Lie algebra $(\mathfrak{G}^{\mathbb{C}})_{\mathbb{R}}$ such that

$$(\mathfrak{G}^{\mathbb{C}})_{\mathbb{R}} = \mathfrak{G} \oplus_{\mathbb{R}} i\mathfrak{G} \qquad \text{direct sum of vector spaces.} \tag{C.2.12}$$

In other words, A real form of a Lie algebra is just a choice of generators for which the structure constants are real. For example, The complex algebra $\mathfrak{sl}(2,\mathbb{C})$ of the complex group $\mathrm{SL}(2,\mathbb{C})$ has two real forms; the compact $\mathfrak{su}(2)$ algebra and the non-compact $\mathfrak{sl}(2,\mathbb{R})$ algebra. The possible third real form $\mathfrak{su}(1,1)$ is included as it is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

Any finite dimensional $\mathfrak{G}^{\mathbb{C}}$ possesses a unique real form in which all the generators are **compact**. Compact means that the scalar product of the generators, defined by the Killing metric, is negative definite. It is given by taking the generators[4]

$$\hat{U}_\alpha = i(E_\alpha + E_{-\alpha}),\ \hat{V}_\alpha = (E_\alpha - E_{-\alpha}),\ \hat{H}_I = iH_I. \tag{C.2.13}$$

We refer to this compact algebra as $\mathfrak{G}^{cp}$.

Definition: An **involution** is a map which is an automorphism defined by

$$\theta(T_a T_b) = \theta(T_a)\theta(T_b) \qquad \forall\, T_a, T_b \in \mathfrak{G}, \qquad \theta^2 = 1. \tag{C.2.14}$$

[4]The compact nature of the generators follows in obvious way from the fact that the only non zero Killing metric between E_α and $E_{-\alpha}$ is $B(E_\alpha, E_{-\alpha}) = 1$ and $B(H_I, H_J) = -(\alpha_I, \alpha_J) < 0$.

By considering all involutions of the unique compact real form $\mathfrak{G}^{cp}$ one can construct all other real forms of $\mathfrak{G}^{\mathbb{C}}$. In particular, the real forms are in one to one correspondence with all those involutive automorphisms of the compact real form [127, 175].

Given an involutive θ we can divide the generators of the compact real form $\mathfrak{G}^{cp}$ into those which possess +1 and -1 eigenvalues of θ. We denote these eigenspaces by

$$\mathfrak{G} = \mathfrak{H} \oplus \hat{\mathfrak{F}} \tag{C.2.15}$$

respectively. Since θ is an automorphism it preserves the structure of the algebra and as a result the algebra when written in terms of this split must take the generic form

$$[\mathfrak{H}, \mathfrak{H}] \subset \mathfrak{H}, \qquad [\mathfrak{H}, \hat{\mathfrak{F}}] \subset \hat{\mathfrak{F}}, \qquad [\hat{\mathfrak{F}}, \hat{\mathfrak{F}}] \subset \mathfrak{H}. \tag{C.2.16}$$

Now, from the generators $\hat{\mathfrak{F}}$ we define new generators $\mathfrak{F} = -i\hat{\mathfrak{F}}$, whereupon the algebra now takes the generic form

$$[\mathfrak{H}, \mathfrak{H}] \subset \mathfrak{H}, \qquad [\mathfrak{H}, \mathfrak{F}] \subset \mathfrak{F}] \subset \mathfrak{F}, \qquad [\mathfrak{F}, \mathfrak{F}] \subset (-1)\mathfrak{H}. \tag{C.2.17}$$

Thus we find a *new* real form of $\mathfrak{G}^{\mathbb{C}}$ in which the generators $\mathfrak{H}$ are compact while the generators $\mathfrak{F}$ are **non-compact**[5]. Clearly, the new real form has **maximal compact subalgebra** $\mathfrak{H}$ and this is just the part of the algebra invariant under θ.

As each real form corresponds to an involutive θ we can write the corresponding real form as $\mathfrak{G}_\theta$[6]. The number of compact generators is dim$\mathfrak{H}$ and the number of non-compact generators is dim$\mathfrak{G}$ − dim$\mathfrak{H}$.

Definition: The **character** σ of the real form is the number of non-compact minus the number of compact generators and so $\sigma = \dim\mathfrak{G} - 2\dim\mathfrak{H}$.

If the involutive θ is taken to be θ_c which is a linear operator that takes $E_\alpha \leftrightarrow -E_{-\alpha}$ and $H_I \to -H_I$, an important real form can be constructed. Accordingly, the generators of the compact real form transform as $\hat{V}_\alpha \to \hat{V}_\alpha$, $\hat{U}_\alpha \to -\hat{U}_\alpha$, and $\hat{H}_I \to -\hat{H}_I$ where $\hat{V}_\alpha = E_\alpha - E_{-\alpha}$, $\hat{U}_\alpha = E_\alpha + E_{-\alpha}$ and $\hat{H}_I = H_I$. Using θ_c we find a real form with generators

$$V_\alpha = \hat{V}_\alpha, \qquad U_\alpha = -i\hat{U}_\alpha, \qquad H_I = -i\hat{H}_I. \tag{C.2.18}$$

The V_α remain compact generators while U_α and H_I become non-compact[7]. Clearly, the non-compact part of the real form of the algebra found in this way contains all the Cartan subalgebra CSA and it turns out that it has the maximal number of

[5]This follows from the fact that all the generators in the original algebra are compact and so have negative definite Killing metric and as a result of the change all the generators $\mathfrak{F}$ will have positive definite B.

[6]For the compact real form the involution is just the identity map Id on all the generators and so we may write $\mathfrak{G}^{cp} = \mathfrak{G}_I$.

[7]We are denoting with H_I both the Cartan generators of $G^{\mathbb{C}}$ and the Cartan generators in this particular real form. The maximal compact subalgebra is just that invariant under θ_c.

non-compact generators of all real forms one can construct. It is therefore called the **maximally non-compact** real form or split real form[8] denoted by $\mathfrak{G}_{\theta_c}$. Let's consider two examples:

(1) The complex Lie algebra $\mathfrak{G}^{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C})$ has $\mathfrak{su}(n,\mathbb{C}) = \mathfrak{G}^{cp}$ as its unique compact real form and $\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{G}_{\theta_c}$ as its maximally non-compact real form.

(2) For $\mathfrak{e}_8$ algebra of group E_8 the maximally non-compact real form is denoted by $\mathfrak{e}_{8(8)} = \mathfrak{G}_{\theta_c}$ and its maximal compact subalgebra is $\mathfrak{so}(16)$ of group SO(16). The character of $\mathfrak{e}_{8(8)}$ is $\sigma = 248 - 2.120 = 8 = \mathrm{rank}(E_8)$. This notation may use for all the exceptional groups.

Taking different non-trivial involutions we find different real forms. For example, for the real form of E_8 denoted by $\mathfrak{e}_{8(-24)}$ the maximal compact subalgebra is $\mathfrak{e}_7 \otimes \mathfrak{su}(2)$.

As the involution θ is an automorphism it preserves the Killing metric and as a result

$$B(\theta(X), \theta(Y)) = B(X,Y) = -B(X,Y) = 0 \text{ if } X \in \mathfrak{H},\ Y \in \mathfrak{F}. \tag{C.2.19}$$

Thus the spaces $\mathfrak{H}$ and $\mathfrak{F}$ are **orthogonal**[9]. As one can realize from the previous discussion the Cartan subalgebra CSA $\mathfrak{h}$ of $\mathfrak{G}_\theta$ can be split between compact generators $\mathfrak{H}$ and non-compact generators of $\mathfrak{F}$. Let us denote the Cartan subalgebra elements in $\mathfrak{F}$ by $\mathfrak{c} = \mathfrak{h} \cap \mathfrak{F}$. The real rank r_θ of $\mathfrak{G}_\theta$ is the dimension of $\mathfrak{c}$. Clearly, it takes its maximal value for maximally non-compact case where it equals the rank r of $\mathfrak{G}_\theta$.

[8] In some literatures the involution θ is called the Cartan involution, and the involution corresponds to the split form $\mathfrak{G}_{\theta_c}$ is called the Chevalley involution θ_c.

[9] It also follows from this discussion that $B(X, \theta(Y))$ is negative definite. In fact one can define a Cartan involution for which this true.

Appendix D

Publications

- [A] D. B. Westra and W. Chemissany, *Coset symmetries in dimensionally reduced heterotic supergravity*, JHEP **0602** (2006) 004 [hep-th/0510137].
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